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INCOMPRESSIBLE POTENTIAL FLOW ABOUT AXIALLY SYMMETRIC DUCTED BODIES

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ABSTRACT

The velocity potentials and stream functions for a point source, doublet, ring source, ring doublet, disk source, disk doublet, ring vortex and line source are derived for the case of axially symmetric incompressible potential flow in an infinitely long duct of constant diameter.

Methods for the development of arbitrary body shapes in uniform ducted flow are presented. Several practical applications are indicated.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

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NOMENCLATURE

a — radius of a ring or disk source, sink or doublet

e — natural logarithm base; $e = 2.71828$

ψ — stream function

I, K — modified Bessel functions of the first and second kind respectively

j — eigenvalue

J, Y — Bessel function of the first and second kind respectively.

m — strength of a source or sink form

M — strength of a doublet form

r — dimensionless radial coordinate; ratio of radial distance to duct radius.

S — ratio of $2m/U$

U — superimposed uniform flow velocity

z — dimensionless axial coordinate; ratio of axial distance to duct radius

ϕ — velocity potential

ϕ_z, ϕ_r — axial and radial velocity components respectively

$| \ |$ — absolute value sign

ψ^* — $|\psi|/2m$

ϕ_z^* — $|\phi_z|/2m$

ϕ_r^* — $\phi_r/2m$

Subscripts:

r — partial differentiation with respect to (r)

z — partial differentiation with respect to (z)

n, s, ν — consecutive numbers; 1, 2, 3, 4, 5, . . .

INTRODUCTION

The problem of determining the complete flow field about axially symmetric ducted bodies occurs widely. Typical instances are the design and analysis of the performance of axisymmetric nozzles and diffusers containing center bodies, axisymmetric wind tunnel test arrangements, ring type flame holders, and turbomachinery elements generally. In view of the numerous practical applications, it was decided to initiate a study of the problem for the case of incompressible flow.

Little theoretical work has been published concerning the flow about ducted axisymmetric bodies, especially for the case where the body fills a sizable portion of the cross sectional area of the duct. The present theoretical study was well advanced before it was discovered that a solution for a point source in an infinitely long duct of constant diameter had recently been published (Ref. 1). A numerical error occurs, however, in the expression for the velocity potential of a source as given there. The expression is rederived here, as it forms the basis for the development of a general approach to the study of the flow past ducted axisymmetric bodies.

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ANALYSIS

The problem to be considered is the case of steady, incompressible and potential flow about axisymmetric bodies located on the axis of an infinitely long, circular tube of constant diameter. The coordinate system and the [illegible] notation are indicated in Fig. 1.

The differential equation for the velocity potential of an axisymmetric incompressible flow is well known, and may be written as (Ref. 2),

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0 \tag{1}$$

Fortunately, this equation is amenable to solution by the method of separation of variables. Further, Equation 1 is linear, so that the principle of superposition can be utilized to build up complex solutions by merely adding simple solutions together.

1. Point Source on the Tube Axis

An expression for the velocity potential of a point source will be developed first. Once it is in hand, the potential solutions for a sink and a doublet follow directly. These solutions may be superimposed on each other to construct axisymmetric bodies of the desired shape.

Consider a source, of strength (m), as shown in Fig. 2, located at r=z=0. The boundary conditions which must be satisfied are:

I. The source flow is uniform at $z = \pm\infty$. Thus,

$$(\phi_r)_{z=\pm\infty} = 0$$

II. At $z = +\infty$, the velocity of the source (2m), and at $z = -\infty$, the velocity of the source flow is (−2m). This follows from continuity as the total source flow splits in half about the (r) axis. The (±) sign accounts for the velocity direction as indicated in Fig. 1.

III. The radial component of the velocity is zero at the duct wall. Thus,

$$(\phi_r)_{r=a} = 0$$

IV. The source flow is symmetric about the (r) axis as well as the (z) axis. However, a singularity exists at r=z=0. Hence, $(\phi_z)_{z=0} = 0$ for $r \neq 0$. Proceeding by the method of separation of variables, let

$$\phi = R(r)\,Z(z) \tag{2}$$

and substituting into Eq. 1, yields

$$\frac{1}{R} R_{rr} + \frac{1}{rR} R_r = -\frac{1}{Z} Z_{zz} \tag{3}$$

[illegible] (r) and (z) are independent, the only possible solutions to Eq. [illegible] is

$$\frac{1}{R} R_{rr} + \frac{1}{rR} R_r = -\frac{1}{Z} Z_{zz} = k = \text{constant} \tag{4}$$

There exist three possibilities: $k = 0$, $k = j^2$, $k = -j^2$, where (j) is a constant.

Examining the case of $k = -j^2$, $j \neq 0$, one finds

$$Z_{zz} - j^2 Z = 0$$

$$R_{rr} + \frac{1}{r} R_r + j^2 R = 0$$

The solutions to the above equations are well known, as

$$[illegible]$$

and

$$R = E_1 J_0(jr) + E_2 Y_0(jr)$$

respectively, where (J_0) is the Bessel function of the first kind of zero order, and (Y_0) is the Bessel function of the second kind of zero order. Therefore, a solution to Eq. 1 exists in the form

$$\phi = (D_1 e^{-jz} + D_2 e^{jz})\,(E_1 J_0(jr) + E_2 Y_0(jr))$$

To satisfy conditions I and II, it is apparent that [illegible] and either [illegible] depending on whether (z) is positive or negative. Hence, one can write,

Further, to satisfy condition II, the solution must have the form,

$$\phi = [illegible]$$

where the term ($2m$) can be thought of as a solution to Eq. 1, for the case of k = 0.

The condition III, requires that

$$J_1(j) = 0$$

Therefore, the values of (j) are the nth-ordered roots of (J_1), where (J_1) is the Bessel function of the first kind of first order. The form of the solution then becomes,

$$\phi_n = 2m + A_n e^{-j_n|z|} J_0(j_n r) \qquad (5)$$

There is now a solution corresponding to each value of (j). It should be noted that conditions I, II and III have been fulfilled. To satisfy condition IV, a series solution can be built up, using Eq. 5, whereby,

$$\phi = 2m + \sum_{n=1}^{\infty} A_n e^{-j_n|z|} J_0(j_n r)$$

Introducing condition IV into Eq. 6, one has,

$$(\phi_z)_{z=0} = 2m - \sum_{n=1}^{\infty} A_n j_n J_0(j_n r)$$

The orthogonality property of Bessel functions is such that (Ref. 3)

$$\int_0^1 r\, J_0(j_n r)\, J_0(j_m r)\, dr = \begin{cases} 0 & n \neq m \\ f(j) & n = m \end{cases}$$

Using the above property, one obtains

$$\int_0^1 r J_0(j_m r)\, (\phi_z)_{z=0} dr = 2m \int_0^1 r J_0(j_m r) dr - A_m j_m \int_0^1 r J_0^2 (j_m r) dr$$

The second integral on the right can be readily evaluated (Ref. 3) as;

$$A_m j_m \int_0^1 r J_0^2 (j_m r) dr = \frac{A_m j_m}{2} J_0^2 (j_m)$$

The first integral on the right is zero, as

$$\int_0^1 rJ_0(\lambda_m r)dr = \frac{1}{\lambda_m}\left[J_1(\lambda_m r)\right]_0^1 = 0$$

Thus,

$$A_m = \frac{-2}{\lambda_m J_0^2(\lambda_m)}\int_0^1 (\phi_z)_{z=0}\, rJ_0(\lambda_m r)dr$$

Taking the size of the source to be very small (close to a mathematical point), one can write, in accordance with Condition IV,

$$A_m = \frac{-2}{\lambda_m J_0^2(\lambda_m)}\lim_{r\to 0}\int_0^r (\phi_z)_{z=0}\, rJ_0(\lambda_m r)dr$$

But, $\lim\limits_{r\to 0} J_0(\lambda_m r) = 1$, so that one has,

$$A_m = \frac{-2}{\lambda_m J_0^2(\lambda_m)}\cdot\frac{1}{2\pi}\lim_{r\to 0}\int_0^{2\pi}\int_0^1 (\phi_z)_{z=0}\, r dr d\theta$$

The integral above represents the flow emitted by the source, which is $(2\pi m)$ in either direction by Gauss's theorem, so that

$$A_m = \frac{-2m}{\lambda_m J_0^2(\lambda_m)}$$

A solution to Eq. 1, satisfying the four boundary conditions can now be obtained by substituting the equation for the coefficients (A_m) into Eq. 6, yielding,

$$\phi = \mp mz - 2m\sum_{m=1}^{\infty}\frac{e^{-\lambda_m |z|}}{\lambda_m J_0^2(\lambda_m)}J_0(\lambda_m r) \tag{7}$$

Returning to Eq. 4, and examining the case of $K = j^2$, one finds that the boundary conditions cannot be satisfied with a series type of solution. Details of this case are presented in Appendix I.

The stream function corresponding to the above velocity potential is

$$\Psi = \pm mr^2 \mp 2mr \sum_{n=1}^{\infty} \frac{e^{-j_n|z|}}{j_n J_0^2(j_n)} J_1(j_n r) \tag{8}$$

sign convention: (+) for $z > 0$
(-) for $z < 0$

At $z = 0$, the stream function is discontinuous, so that approaching $z = 0$ from the left (see Fig. 2), $\Psi = -m$, while approaching $z = 0$ from the right, $\Psi = +m$.

That Eq. 8 represents the stream function may be seen by direct substitution into the following relationships for axisymmetric, incompressible, potential flow,

$$\phi_r = -\frac{1}{r}\Psi_z \qquad \phi_z = \frac{1}{r}\Psi_r \tag{9}$$

The solution for a sink follows directly, as it influences the derived velocity potential and stream function only through the sign (±) associated with the strength (m).

The axial and radial velocity components can be determined by the appropriate partial differentiation of Eq. 7, so that,

$$\phi_z = \pm 2m \pm 2m \sum_{n=1}^{\infty} \frac{e^{-j_n|z|}}{J_0^2(j_n)} J_0(j_n r) \tag{10}$$

sign convention: (+) for $z > 0$
(-) for $z < 0$

$\phi_z = 0$ at $z = 0$

$$\phi_r = 2m \sum_{n=1}^{\infty} \frac{e^{-j_n|z|}}{J_0^2(j_n)} J_1(j_n r) \tag{11}$$

To calculate the streamlines and velocity components, it is convenient to introduce,

$$\psi^* = \frac{\psi}{2m}, \quad \phi_z^* = \frac{1}{2m}|\phi_z|, \quad \phi_r^* = \frac{1}{2m}|\phi_r|$$

Fortunately, the values of $J_0(\beta r)$ and $J_1(\beta r)$ over the range r = 0(.02)1 have been tabulated in Reference 4, for the first ten roots of J_1. After the first ten roots have been used, the difference in the roots is close enough to π so that additional terms can be rapidly evaluated by using the asymptotic approximation given in References 5 and 6,

$$J_n(V) \xrightarrow[V \to \infty]{} (2/\pi V)^{1/2} \cos(V - n\pi/2 - \pi/4) \qquad (12)$$

For values of V > 15, the above relationship is in agreement with the exact value to within one percent, (ref. 7). For larger values of V, the approximation becomes more accurate. Values of ψ^*, ϕ_z^*, ϕ_r^* were calculated on a Remington Rand 1103 computer at the Aeronautical Research Laboratory of the Wright Air Development Center. The results are presented in Tables I, II, III. Sufficient values are included in those tables to facilitate the analysis of flow about bodies of arbitrary shape.

The asymptotic relationship was used to calculate enough terms to assure convergence to five decimal places for the values of .3 ≤ r ≤ 1, z = .1, and z = .2. The remaining values are tabulated for the first ten roots of J_1 only. For z ≥ .3, the first ten roots were sufficient to obtain convergence to five decimal places. For values of z > 1, only one or two terms are necessary, hence the tables were restricted to values of z ≤ 1. Scientific notation is used throughout the tables, so that the number and sign proceeding the tabulated values locate the decimal place. For example, 7.4658-2 may be written as .074658.

For a point source not located at the origin, one has only to consider that (z) (in Eqs. 8, 10, 11) represents the distance along the (z) axis from the plane of the source to the point where the velocity or stream function is being evaluated. In this regard, the absolute value signs |z| were dropped throughout the tables for convenience.

2. Doublet on the Tube Axis

To derive the velocity potential for a doublet, one can consider a sink at the origin and a source of the same strength at z = Δz, r = 0, as shown in Fig. 3. The expressions for the source and sink follow from the development above.

$$\phi_{so} = 2m(z - \Delta z) - 2m \sum_{n=1}^{\infty} \frac{e^{-j_n(z - \Delta z)}}{j_n J_o^2(j_n)} J_o(j_n r)$$

$$\phi_{si} = -2mz + 2m \sum_{n=1}^{\infty} \frac{e^{-j_n(z)}}{j_n J_o^2(j_n)} J_o(j_n r)$$

At the point (P), the velocity potential due to the source and sink, taken in the limit as $\Delta z \rightarrow 0$, is

$$\phi = \lim_{\Delta z \rightarrow 0}(\phi_{so} + \phi_{si}) = \lim_{\Delta z \rightarrow 0} 2m \sum_{n=1}^{\infty} \frac{e^{-j_n z}}{j_n J_o^2(j_n)} J_o(j_n r)\,(1 - e^{j_n \Delta z}) - 2m\Delta z$$

or, expanding,

$$\phi = \lim_{\Delta z \rightarrow 0} 2m \sum_{n=1}^{\infty} \frac{e^{-j_n z}}{j_n J_o^2(j_n)} J_o(j_n r)\left(-j_n \Delta z - \frac{(j_n \Delta z)^2}{2} - \frac{(j_n \Delta z)^3}{3!} \cdots\right) - 2m\Delta z$$

The strength of the doublet is defined as.

$$M = \lim_{\Delta z \rightarrow 0} m \Delta z$$

Thus, only the first term remains in the expansion above, so that the velocity potential for a doublet located at the origin is,

$$\phi = -2M \sum_{n=1}^{\infty} \frac{e^{-j_n |z|}}{J_o^2(j_n)} J_o(j_n r) - 2M \qquad (13)$$

$$\psi = 2Mr \sum_{n=1}^{\infty} \frac{e^{-j_n |z|}}{J_o^2(j_n)} J_1(j_n r) \qquad (14)$$

One should note the sign convention adopted above for the doublet, for as derived, the axis of the doublet lies on the (z) axis and the doublet flow is counter clockwise above the (z) axis.

The streamlines for a doublet are shown in Figure 4. The calculations for the doublet were carried out in much the same manner as the case for the source streamlines. However, Eq. 14 converges much more slowly than does Eq. 8, requiring a greater number of terms.

In addition, the limit of Eq. 8 is known at $z = 0$, by symmetry and continuity, to be $\Psi = m$, but at $z = 0$, Eq. 14 is more difficult to evaluate. From the nature of the doublet flow, the slope of the streamlines must be zero at $z = 0$, hence, Eq. 14 was evaluated for values of (z) close to zero, and then the streamlines were faired in. To estimate the value of Eq. 14 at $z = 0$, slopes of the curves corresponding to Eq. 8 and 9 were found graphically at $z = 0$.

3. Ring Source

The velocity potential of a ring source can be found using the same line of attack as used for the simple source. The boundary conditions put forth for the point source are valid in this case also, except for condition IV, which must be modified as follows:

IV(a) The source flow is symmetric about the (r) axis. However, the ring $r = a$ is a singularity. Thus,

$$(\phi_z)_{z=0} = 0 \text{ for } r \neq a$$

The total strength of the ring, which is taken to be uniformly distributed around the circumference, is (m), so that the total source flow emitted by the ring is $(4\pi m)$. Therefore, Eq. 6 is valid for a ring source as well as a point source. To obtain the coefficients (A_n), the condition must be satisfied that,

$$A_n = \frac{-2}{j_n J_0^2(j_n)} \lim_{\epsilon \to 0} \int_{a-\epsilon}^{a+\epsilon} (\phi_z)_{z=0}\, r\, J_0(j_n r)\, dr$$

But this may be written as,

$$A_n = \frac{-2}{j_n J_0^2(j_n)} J_0(j_n a) \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_0^{2\pi}\int_{a-\epsilon}^{a+\epsilon} (\phi_z)_{z=0}\, r\, dr\, d\theta$$

The integral now represents the flow emitted by the ring source, which is $(2\pi m)$ in either direction, so that,

$$A_n = \frac{-2m}{j_n J_0^2(j_n)} J_0(j_n a)$$

The velocity potential for the ring source follows,

$$\phi = -|z| m - 2m \sum_{n=1}^{\infty} \frac{e^{-j_n |z|}}{j_n J_0^2(j_n)} J_0(j_n a)\, J_0(j_n r) \qquad (15)$$

The corresponding stream function, satisfying Eq. 7, is

$$\Psi = \pm m r^2 \pm 2mr \sum_{n=1}^{\infty} \frac{e^{-j_n|z|}}{j_n J_0^2(j_n)} J_1(j_n r)\, J_0(j_n a) \qquad (16)$$

sign convention: $\begin{Bmatrix} (+) \text{ for } z > 0 \\ (-) \text{ for } z < 0 \end{Bmatrix}$

At $z = 0$, $\Psi = \pm m$, depending on whether the plane of the ring source is approached from the right or left.

A check on the validity of the solution can be made by taking the limit as $(a \to 0)$ so that $(J_0(j_n a) \to 1)$ and the expressions reduce to those found for a point source on the axis.

4. Ring Doublet

The velocity potential for a ring doublet and the corresponding stream function can be obtained by using the same technique as used to obtain the point doublet. The resultant expressions are,

$$\phi = -2M - 2M \sum_{n=1}^{\infty} \frac{e^{-j_n|z|}}{J_0^2(j_n)} J_0(j_n r)\, J_0(j_n a) \qquad (17)$$

$$\Psi = 2Mr \sum_{n=1}^{\infty} \frac{e^{-j_n|z|}}{J_0^2(j_n)} J_1(j_n r)\, J_0(j_n a) \qquad (18)$$

5. Disk Source

The velocity potential of a disk source can be derived using the same boundary conditions as for the case of the point source, except condition IV, which must be modified as,

IV(b) The source flow is symmetrical about the (z) axis. However, the disk of radius $r = a$ is a singularity. Thus,

$$(\phi_z)_{z=0} = 0 \text{ for } r > a$$

The total strength of the doublet, which is taken to be uniformly distributed over the disk, is (m). Returning to Eq. 6, the coefficients (A_n) can be determined by utilizing condition IV(b), such that,

$$A_n = \frac{-2}{j_n J_0^2(j_n)} \int_0^1 (\phi_z)_{z=0} J_0(j_n r)\, r\, dr$$

By continuity, $a^2(\phi_z)_{z=0} = 2m$, so that,

$$A_n = \frac{-2}{i_n J_0^2(i_n)} \quad \frac{2m}{a^2} \int_0^a J_0(i_n r) r\,dr$$

or,

$$A_n = \frac{-4m}{a\, i_n^2 J_0^2(i_n)} J_1(i_n a)$$

The expression for the velocity potential follows.

$$\phi = \pm 2m - \frac{4m}{a} \sum_{n=1}^{\infty} \frac{e^{-i_n|z|}}{i_n^2 J_0^2(i_n)} J_0(i_n r)\, J_1(i_n a) \qquad (19)$$

$$\Psi = \pm mr^2 \pm \frac{4m}{a} \sum_{n=1}^{\infty} \frac{e^{-i_n|z|}}{i_n^2 J_0^2(i_n)} J_1(i_n r)\, J_1(i_n a) \qquad (20)$$

sign convention: (+) for $z > 0$
(−) for $z < 0$

At $z = 0$, $\Psi = \pm m$, depending on whether the plane of the disk source is approached from the right or left. The results can be checked by taking the limit as $(a \rightarrow 0)$, whereupon $J_1(i_n a) \rightarrow \frac{i_n a}{2}$, so that the above expressions reduce to those for a point source.

Comparing Eqs. 20 and 8, it is apparent that the convergence of Eq. 20 will be faster, so that the use of disks for developing bodies of general shape rather than point sources should reduce the amount of computation.

6. Disk Doublet

The velocity potential for a disk doublet can be found by utilizing the technique used for a point doublet. The velocity potential is found to be,

$$\phi = -2M - 2M \sum_{n=1}^{\infty} \frac{e^{-i_n|z|}}{i_n J_0^2(i_n)} J_0(i_n r)\, J_1(i_n a) \qquad (21)$$

The corresponding stream function is,

$$\Psi = \frac{4Mr}{a} \sum_{n=1}^{\infty} \frac{e^{-j_n|z|}}{j_n J_0^2(j_n)} J_1(j_n r)\, J_1(j_n a) \tag{22}$$

7. <u>Ring Vortex</u>

The velocity potential for a ring vortex is readily obtained by employing the property that a ring vortex is equivalent to a uniform distribution of doublets over the surface bounded by it. (ref. 2). The axes of the doublets are taken normal to the surface every where, and the density of the distribution is taken to be equal to the strength of the vortex. Thus, by Eq. 21, one has,

$$\phi = -2\pi a^2 K - 4\pi a K \sum_{n=1}^{\infty} \frac{e^{-j_n|z|}}{j_n J_0^2(j_n)} J_0(j_n r)\, J_1(j_n a) \tag{23}$$

where (K) is the strength of the vortex.

The corresponding stream function is,

$$\Psi = 4\pi a K r \sum_{n=1}^{\infty} \frac{e^{-j_n|z|}}{j_n J_0^2(j_n)} J_1(j_n r)\, J_1(j_n a) \tag{24}$$

8. <u>Line Source</u>

Continuous distributions of sources, sinks and doublets are often used to develop bodies of various shapes. Considering the case of a line source of constant strength per unit length (m_1), and letting (z_1) be the source coordinate, then,

$$d\phi = 2m_1(z-z_1)dz_1 - 2m_1 \sum_{n=1}^{\infty} \frac{e^{-j_n(z-z_1)}}{j_n J_0^2(j_n)} J_0(j_n r)\, dz_1$$

so that the velocity potential is,

$$\phi = 2mz - m(z_a+z_b) - \frac{2m}{z_s} \sum_{n=1}^{\infty} \frac{e^{-j_n|z-z_a|}}{j_n^2 J_0^2(j_n)} J_0(j_n r)\,(e^{j_n z_s}-1) \tag{25}$$

where (z_a) and (z_b) denote the beginning and end of the source respectively, and (z_s) is the length of the source. The corresponding stream function is,

$$\Psi = \frac{Ur^2}{2} - \frac{2mr}{a} \sum_{n=1}^{\infty} \frac{e^{-\lambda_n |z-a_0|}}{\lambda_n^2 J_0^2(\lambda_n)} J_1(\lambda_n r) \left(e^{\lambda_n a_0} - 1\right) \qquad (26)$$

The results can be checked by taking the limit as $a_0 \to 0$ whereupon the expressions reduce to those for a point source. The convergence of the functions above does not appear to be any better than for the case of a point source, so that the additional complexity of the results may not warrant the use of continuous distributions. Linear or exponential variations of the source strength can be treated in the same manner as the above case, but the resultant expressions become increasingly cumbersome.

TYPICAL APPLICATIONS

A. SIMPLE BODIES

The study of bodies of arbitrary and practical shape will be initiated by first considering the simple shapes resulting when single sources, doublets, disk doublets and ring doublets of varying strength are placed in a ducted uniform flow.

1. Point Source in a Uniform Flow

The velocity potential of a point source in a ducted uniform flow is found by adding the velocity potential (Uz) for a uniform flow to Eq. 7. To obtain the corresponding stream function, one has to add the term ($Ur^2/2$) to Eq. 8. In addition, a constant ($2m$) must be subtracted from the stream function when the flow downstream of the source is being considered.

The form of the stream function indicated above stems from the mechanism by which a body streamline is developed when a point source is placed in a uniform flow. The source flow going upstream is turned back by the superimposed flow, so that upstream of the source, there is no net source flow (in the (z) direction) within the body streamlines. However, downstream of the source, both the original downstream source flow and the original upstream source flow which has been turned back downstream now exists (i.e., the total source flow). The usual convention for the stream function, is that a body streamline is indicated when its value is zero, i.e., $\Psi = 0$. In order to retain this convention, it is then necessary to adjust the expression for the stream function downstream of the source by subtracting a constant which accounts for the source flow within the body streamlines. In this case, the constant is ($2m$) as the stream function concerns only the flow in the upper half plane.

Bodies were calculated for various source strengths using the results obtained from the analysis in the manner indicated above. To use the tables at the end of the text, the body streamline is determined by the condition that,

$$r^* = \frac{Ur^2}{4m} \quad \text{for} \quad z < 0$$

$$r^* = \frac{Ur^2}{4m} + 1 \quad \text{for} \quad z > 0$$

The results for several values of (U/m) are shown in Fig. 5.

The radial components of the velocities, made dimensionless with (U), are simply,

$$\phi_r = \frac{2m}{U}\phi_r^*$$

which can be evaluated directly from Table III.

The axial velocity components can be expressed similarly as,

$$\phi_z = \left(1 \pm \frac{2m}{U}\phi_z^*\right)$$

sign convention: $\begin{cases} (+) & \text{for } z > 0 \\ (-) & \text{for } z < 0 \end{cases}$

At $z = 0$, $\phi_z^* = 0$. The axial velocity components can be evaluated by using Table II.

There is a departure from the usual result that the velocity upstream at infinity is just that due to the uniform flow superimposed on the source flow. In this case, the velocity upstream at infinity is less than the superimposed value by (2m). Similarly, the velocity downstream at infinity is greater by (2m).

The radius of the body at the plane of the source can be quickly found by employing the continuity equation, thus,

$$U - 2m = U(1 - r_0^2)$$

so that

$$r_0^2 = \frac{2m}{U}$$

The radius of the body downstream at $z = +\infty$ can also be determined by continuity as,

$$U\left(1 - \frac{r^2}{\ldots}\right) = (U + 2m)\left(1 - \frac{r^2}{\ldots}\right)$$

It follows that,

$$r_\infty^2 = \frac{2\, r_0^2}{1 + r_0^2}$$

The formulae above indicate a criteria for the establishment of flow through the duct, namely, that $(U > 2m)$. In order to establish the stagnation point, one can use Table II to determine at what value of (z),

$$(K_z^0)_{r=0} = \frac{U}{2m}$$

2. Doublet in a Uniform Flow

The velocity potential for a doublet in a uniform flow can be obtained by simply adding the velocity potential (Uz) for a uniform flow to Eq. 13. The corresponding stream function can be obtained by adding $(Ur^2/2)$ to Eq. 14.

The body streamlines can be determined by using Table III, as the stream function can be written as,

$$F = \frac{Ur^2}{2} + 2m\, K_z$$

Bodies calculated for several values of (M/U) are shown in Fig. 6. The calculation of the velocities was not attempted as the convergence close to the plane of the doublet is very poor.

3. Ring Doublet in a Uniform Flow

The velocity potential for a ring doublet in a uniform flow can be obtained by adding the velocity potential (Uz) for a uniform flow to Eq. 17. The corresponding stream function can be obtained by adding $(Ur^2/2)$ to Eq. 18.

Care must be taken to observe the proper sign convention on (M), for to obtain a body, the doublet flow on the (z) axis and upstream of the doublet must be opposed to the direction of (U). Hence, if (U) is positive, (M) is negative.

A body calculated for $M/U = .5$ and $a = .5$, is shown in Fig. 7.

4. Disk Doublet in a Uniform Flow

The velocity potential for a disk doublet in a uniform flow can be obtained by adding the velocity potential (Ux) for a uniform flow to Eq. 21. The corresponding stream function can be obtained by adding ($Ur^2/2$) to Eq. 22.

Bodies calculated for ($M = aU$) and several values of (a) are shown in Fig. 8.

D. PRESCRIBED BODY SHAPE

The velocity and pressure distributions about bodies of special shape are often of practical interest. Numerous methods for the determination of the source and sink distribution corresponding to a non-ducted body occur in the literature (e.g., Ref. 8). Applying similar techniques to the case at hand, the aim is then to evolve a method for determining source and sink distributions which approximate a ducted body shape. Having the source and sink distribution is all that is required to completely define the velocity and pressure distribution about the body.

1. Discrete Distributions of Sources and Sinks

From the analysis of a point source in a uniform flow, it is clear that specifying one coordinate of the body surface determines the strength of the source (m). Consequently, for each specified body coordinate, one must place a source or sink in the flow. Since only axisymmetric bodies are being considered, the sources and sinks are located on the (x) axis. By judiciously choosing the location of these sources and sinks (i.e., close to the (x) coordinate specified or at least within the length of the desired body) one might expect that the body shape between the chosen coordinates will closely approximate the desired shape. To carry this approach out, it is convenient to let

$$\Psi_s = \frac{\psi_s}{2m_s} = \frac{r^2}{2} + r\sum_{n=1}^{\infty} \frac{e^{-j_n|x_s|}}{j_n J_0^2(j_n)} J_1(j_n r) \qquad (27)$$

Then the stream function for the total (s) sources in a uniform flow is,

$$\Psi_t = \frac{r^2}{2} + \sum_{s=1}^{s} a_s(\Psi_s) \qquad (28)$$

where

$$a_s = \frac{2m_s}{U}$$

[illegible] previously, namely (+) when $z > 0$, and (−) when $z < 0$, and at $z = 0$, $F^*_z = 1/2$, and the sign is negative. It was mentioned previously, but will be further emphasized here, that $|z|$ is the absolute value of the axial distance from the source in question to the point where the stream function of velocity potential is being evaluated. The value of $|z|$ is then relative to the source and not dependent on the arbitrary location of the origin of the coordinate system.

A previous analysis of the source flow inside the body streamlines indicated that the condition $F^* = 0$ defines a point on the body only if the net source flow upstream of this point (within the body) is zero. When a net source flow exists upstream of a particular point on the body, the condition

$$F^*_{\pm} = 1/2 \sum_{u=1}^{n} S_u$$

on the body, where (S_u) are the sources and sinks upstream of the body point.

As an example, consider that three body coordinates are specified, with sources and sinks located on the axis at the same values of (z). Then, one has three equations,

$$1/2 r_1^2 - S_1 F^*_{11} - S_2 F^*_{21} - S_3 F^*_{31} = 0$$

$$1/2 r_2^2 - S_1(1 - F^*_{12}) - S_2 F^*_{22} - S_3 F^*_{32} = 0$$

$$1/2 r_3^2 - S_1(1 - F^*_{13}) - S_2(1 - F^*_{23}) - S_3 F^*_{33} = 0$$

The subscripts are chosen to correspond to the coordinates of the sources, (z_1, z_2, z_3) and the body coordinates $(r_1, z_1; r_2, z_2; r_3, z_3)$. The first subscript on F^* refers to the source location, while the second indicates the body coordinate. The above relationships can be readily put into matrix form.

$$\begin{pmatrix} F^*_{11} & F^*_{21} & F^*_{31} \\ (1-F^*_{12}) & F^*_{22} & F^*_{32} \\ (1-F^*_{13}) & (1-F^*_{23}) & F^*_{33} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = 1/2 \begin{pmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \end{pmatrix}$$

Several observations can be made on the above matrix, which permit one to write the matrix corresponding to any number of prescribed body coordinates in almost mechanical fashion. First, the values on the diagonal of the matrix must be equal to (1/2). Second, to the right of the diagonal are the sources and sinks [illegible] of the body points considered. Third, to the left of the diagonal are the sources and sinks upstream of the body points considered.

Having established a method for obtaining the values of (S_s), one is in a position to determine the complete body form, and also the velocities any where in the surrounding flow field.

The axial velocity component can be found conveniently by letting,

$$(\phi_z^*)_s = \frac{(\phi_z)_s}{2\pi} = 1 + \sum_{n=1}^{\infty} \frac{e^{-j_n|z_s|}}{J_0^2(j_n)} J_0(j_n r) \tag{29}$$

The axial velocity at any point is then,

$$(\phi_z^*)_t = 1 + \sum_{s=1}^{s} S_s(\pm\phi_z^*)_s \tag{30}$$

where the sign is (+) for $z > 0$, and (-) for $z < 0$.

At $z = 0$, $\phi_z = 0$.

Solving Eq. 30 for $(\phi_z^*)_t = 0$ when $r = 0$, yields the location of the stagnation point.

The radial velocity component can be determined similarly, by letting,

$$(\phi_r^*)_s = \sum_{n=1}^{\infty} \frac{e^{-j_n|z_s|}}{J_0^2(j_n)} J_1(j_n r) \tag{31}$$

Thus, the radial velocity at any point is,

$$(\phi_r^*)_t = \sum_{s=1}^{s} S_s(\phi_r^*)_s \tag{32}$$

It may be desired to specify the total strength of the sources and sinks as an initial condition. This can be done conveniently by specifying the radius of the body at $z = +\infty$. Then by continuity,

$$1 - \sum_{s=1}^{s} S_s = (1 + \sum_{s=1}^{s} S_s)(1 - r_\infty^2)$$

so that

$$\sum_{s=1}^{s} S_s = \frac{r_\infty^2}{2 - r_\infty^2}$$

The examples were carried out as above with seven coordinate points and the total source-sink strength specified. The details are presented in Appendix II, and the results in Fig. 9(a) and (b).

2. Continuous Distributions of Source and Sink Discs

Clearly, one could obtain generalized body shapes using many types of source and sink distributions. However, the utilization of source and sink discs appears to offer an advantage in that the series expressions converge more rapidly than for the case of point sources and sinks.

The use of this method is similar to the case where sources and sinks were used.

C. PRESCRIBED BODY COORDINATES AND VELOCITY DISTRIBUTION

Discrete Distribution of Sources and Sinks

The combined problem of specifying certain points on a body and certain flow conditions, and then solving for the complete body shape is often of interest. The technique for solving this problem is similar to that employed when prescribing the body shape alone.

A source or sink must be included for each point or flow condition specified. The appropriate equation relating these conditions is either 28, 30 or 32. Finally, one has a set of (n) linear equations in (n) unknowns to be solved. Upon their solution, the complete body shape can be determined using the methods described in the previous sections. An example is carried out in Appendix III. The results are shown in Fig. 10.

D. INFLUENCE OF LOCAL CURVATURE

Often in the design of ducted bodies, a question arises as to the influence of local body curvature on the adjacent velocity distributions. Two methods of varying the local body curvature were studied.

The first method studied, involved the calculation of bodies using eight specified body coordinates and point sources and sinks on the axis. By maintaining some of the body points the same but varying others slightly, similar bodies with different (localized) curvatures were obtained. The results of this approach are shown in Fig. 9(a). The velocities were calculated at $r = 0$, and $r =$ [illegible]. The results are compared in Fig. 9(b). It can be seen that while the curvature effects were strong on the axial and radial components near the body, the effect on the magnitude of the resultant (meridional)

[illegible] the [illegible]. The changes in radial and axial velocity components apparently diminish each other. The effect of the curvature [illegible] becomes negligible at approximately half the distance from the body to the duct wall.

A second method tested, was the use of a ring source in combination with a point source. The ring and point source were placed in the same plane, with the radius of the ring being close to that desired for the body. While it would have been more desirable to superimpose a ring doublet on a point source, so that there would be no change in the mass flow, the computation of the ring doublet is excessively cumbersome due to the slow rate of convergence. Using a ring and point source, one can prescribe the closeness of the ring to the body surface arbitrarily. This follows from the boundary conditions and continuity. Since it is known that in the plane of a point and ring source, there is no axial velocity component due to the source flow, the axial velocity in this plane is wholly that of the superimposed uniform flow. By continuity, the radius of the body in the plane of the sources is,

$$r_o^2 = \frac{2}{\pi}(m_s + m_r)$$

where (m_s) and (m_r) are the strengths of the point and ring sources respectively. Consequently, when the total source strengths are specified, the body radius is known in the plane of the sources and the radius of the ring can be placed as close to the body surface as desired.

[illegible] studies were made on two examples using the above method. The initial calculation assumed a value of $m_r = .05\ m_s$, which turned out to be too small a value for the ring source, as a negligible effect was had on the body shape. When the strength of the ring source was increased to $m_r = .10\ m_s$, significant changes in body contour occurred. The effect of this change in body contour on the velocity distribution along the duct wall was calculated, and the results are shown in Fig. 11.

CONCLUDING DISCUSSION

[illegible] the primary emphasis has been placed on developing the velocity potential and stream function for various source and sink distributions inside ducts and associated methods for determining special body contours, considerable insight into the general nature of the flow about axially symmetric ducted bodies has been obtained. It is obvious from the nature of the solutions, however, that [illegible] [illegible] computation is required for fully detailed flow studies.

The development of the velocity potential for a point source [illegible] (m). Thus the undisturbed velocity is not equal to the superposed uniform velocity as in the case of a point source in [illegible] uniform flow, but rather $(U + 2m)$ for upstream and $(U - 2m)$ for downstream.

An interesting finding, useful for comparing ducted and non-ducted flows, is that the body radius in the plane of the source (dealing here with bodies due to a single point source) is the same, i.e., $r_0^2 = 2m/U$ in both cases. For the body radius at $x = \infty$, however, the non-ducted case has $r_\infty^2 = 4m/U$, as compared with the ducted case

$$r_\infty^2 = \frac{4m/U}{1 + 2m/U}.$$

One additional point, the location of the stagnation point, will be compared for the two cases. In the case of the non-ducted body (for a single point source), it occurs at $x = -(m/U)^{1/2}$, whereas in the ducted case, it may be located by writing Eq. 92 in the form, $2m/U = 1/G_s$. Using Table II, for $r = 0$, one obtains the following comparison:

stagnation point ducted body	stagnation point, non-ducted body	m/U
-0.10	-0.110	0.0120
-0.20	-0.226	0.0510
-0.30	-0.295	0.0865
-0.40	-0.382	0.1455
-0.50	-0.458	[illegible]
-0.60	-0.522	[illegible]
-0.70	-0.575	[illegible]
-0.80	-0.612	0.3750
-0.90	[illegible]	[illegible]
-1.00	-0.661	0.4960

In conclusion, it is seen that non-ducted and ducted point source bodies are essentially the same for values of (m/U) of about 0.1 or smaller. The corresponding results for a doublet, indicate that the body shape is not appreciably [illegible] [illegible] greater than $(U/3)$.

[illegible] functions, it is apparent that differentiation with respect to either [illegible] [illegible] [illegible] [illegible] to obtain an accurate evaluation of

the series. Consequently, the use of doublets (ring and disk included) for the study of flow about bodies is particularly difficult. This [illegible] the [illegible] expressions for velocity potential of a doublet is actually the result of differentiation of the corresponding velocity potential for a source. Thus, the series for the evaluation of the velocity components about a body due to a doublet in a uniform flow is difficult to compute. For the reasons indicated above, the study of doublet forms was restricted.

In the ring doublet investigated (see Fig. 7), the shape of the inner and outer surfaces is quite different due to the nature of the ring doublet flow. It is likely that this shape difference would diminish if the strength of the ring doublet were decreased. Thus, it appears that the use of ring doublets, sources and sinks may be a practical way of studying the flow about ring type flame holders. Of course high speed computing devices would be required to conduct a detailed study on this problem.

The investigation of the disk doublet in a uniform flow was aided by the fact that the expressions for the velocity potential for a disk doublet converges better than the velocity potential of the other doublet forms studied. For the cases which were calculated, it appears that the resultant body, (when the disk doublet is placed in a uniform flow) is very nearly a circle for body diameters up to one-half the duct diameter. For larger bodies, the shape becomes progressively blunter.

The computation of bodies of arbitrary shape is considerably reduced once the stream function and velocity components for a point source have been evaluated and tabulated. Tables I, II and III contain a sufficient range of these quantities to determine the complete flow field about a wide variety of body shapes. For rapid and approximate studies, graphical methods may be used. For detailed and precise studies, the computation can be set up on an automatic computer in a number of ways.

The influence of local body curvature is of practical interest in the design of the hub of a turbomachine. The shape of the hub influences the blade loading, at least near the hub. An important design problem is how far away, spanwise from the hub, can the blade loading be affected by introducing a change in local curvature of the hub. Another problem is that of obtaining a particular velocity distribution at inlet to the machine which might be more desirable. [illegible] the shape of the hub influences the velocity distribution. The results shown in Figure [illegible] [illegible] changes in hub contour are ineffectual in bringing about significant changes in the velocity distribution.

[illegible] was calculated for a [illegible] velocity distribution. Several points were specified as body coordinates also, so that the basic form of the hub would not be completely different. The result, shown in Fig. 10, indicates, in a preliminary way at least, that the method of prescribing the velocity distribution leads to large fluctuations in body contour. Further, one concludes that the type of velocity distributions obtained in Fig. 9(b) are most likely to [illegible] practical body shapes.

TABLE I

Stream Function F For a Point Source

Z	r = 0.10	r = 0.16	r = 0.20	r = 0.24
0.10	1.[illegible]	2.9[illegible]-1	2.7464575-1	3.0651375-1
0.20	5.[illegible]	1.[illegible]-1	1.[illegible]-1	1.[illegible]-1
0.30	2.[illegible]-2	6.[illegible]-2	8.8682731-2	1.1654158-1
0.40	1.6469951-2	3.9702142-2	5.[illegible]-2	8.0167258-2
0.50	1.1588615-2	2.8609707-2	4.3312837-2	6.0142390-2
0.60	9.0023358-3	2.2525394-2	3.4493682-2	4.8517014-2
0.70	7.5196402-3	1.9969449-2	2.9254934-2	4.1484853-2
0.80	6.5244486-3	1.6776965-2	2.6021529-2	3.7093772-2
0.90	6.[illegible]-3	1.[illegible]-2	2.3945900-2	3.[illegible]-2
1.00	5.7654940-3	1.4585506-2	2.1641978-2	3.[illegible]-2

Z	r = 0.30	r = 0.34	r = 0.38	r = 0.40
0.10	3.4559576-1	3.5351522-1	3.7309714-1	3.[illegible]-1
0.20	2.2987228-1	2.5590553-1	2.7869358-1	2.[illegible]-1
0.30	1.5719956-1	1.8298709-1	2.0749902-1	2.1924607-1
0.40	1.1400471-1	1.3709759-1	1.6018088-1	1.7164549-1
0.50	8.5370426-2	1.0861264-1	1.2963409-1	1.4035317-1
0.60	7.2810089-2	9.0005893-2	1.0997629-1	1.1994220-1
0.70	6.3114890-2	7.9485812-2	9.7238420-2	1.0658954-1
0.80	5.6942617-2	7.[illegible]-2	8.8906674-2	9.7796102-2
0.90	5.2944571-2	6.7406509-2	8.3407250-2	9.1963994-2
1.00	5.0920054-2	6.4251570-2	7.9749296-2	8.8071499-2

Z	r = 0.44	r = 0.48	r = 0.50	r = 0.52
0.10	3.97[illegible]-1	4.0763998-1	4.1236074-1	4.1[illegible]-1
0.20	3.0894143-1	3.2668132-1	3.3498703-1	3.4296977-1
0.30	2.4176673-1	2.6308776-1	2.7394220-1	2.8335241-1
0.40	1.9433966-1	2.1664596-1	2.2765492-1	2.3856850-1
0.50	1.6309974-1	1.8415171-1	1.9526421-1	2.0642520-1
0.60	1.4053584-1	1.6190512-1	1.7284504-1	1.8925308-1
0.70	1.2616508-1	1.4681827-1	1.5752143-1	1.6846984-1
0.80	1.1657998-1	1.3661887-1	1.4708504-1	1.5786985-1
0.90	1.1015175-1	1.2972504-1	1.4001546-1	1.5068433-1
1.00	1.0589591-1	1.2506070-1	1.3520941-1	1.4571025-1

Z	r = 0.54	r = 0.56	r = 0.58	r = 0.60
0.10	4.2215459-1	4.2530698-1	4.9645154-1	4.3319588-1
0.20	3.5066592-1	3.5810601-1	3.6571157-1	3.7239714-1
0.30	2.9313934-1	3.0272482-1	3.1215192-1	3.2137745-1
0.40	[illegible]	2.6013570-1	2.7080571-1	2.8140957-1
0.50	2.17[illegible]-1	2.2[illegible]-1	2.4017212-1	2.5150971-1
0.60	1.9520[illegible]-1	2.0[illegible]-1	2.1826958-1	2.29[illegible]-1
0.70	1.7969810-1	1.9102317-1	2.0265988-1	2.1449709-1
0.80	1.6820997-1	1.8025504-1	1.9186916-1	2.0975540-1
0.90	1.6157679-1	1.7285854-1	1.8441590-1	1.9690574-1
1.00	1.5655962-1	1.6775428-1	1.7929138-1	1.9126847-1

TABLE V – continued

z	r = 0.62	r = 0.64	r = 0.66	r = 0.68
0.10	4.3693999-1	4.4059960-1	4.4408949-1	4.4742845-1
0.20	3.7927844-1	3.8586772-1	3.9242632-1	3.9867452-1
0.30	3.3048540-1	3.3947298-1	3.4835709-1	3.5715759-1
0.40	2.9296157-1	3.0247202-1	3.1295988-1	3.2341715-1
0.50	2.6289622-1	2.7433635-1	2.8583606-1	2.9740288-1
0.60	2.4168590-1	2.5365949-1	2.6577666-1	2.7804042-1
0.70	2.2655140-1	2.3881865-1	2.5129929-1	2.6399425-1
0.80	2.1591180-1	2.2839569-1	2.4102853-1	2.5398721-1
0.90	2.0859558-1	2.2101345-1	2.3382809-1	2.4694875-1
1.00	2.0398359-1	2.1599494-1	2.2882152-1	2.4204249-1

z	r = 0.70	r = 0.72	r = 0.74	r = 0.76
0.10	4.5076599-1	4.5406764-1	4.5735066-1	4.6061964-1
0.20	4.0523151-1	4.1151529-1	4.1774266-1	4.2392916-1
0.30	3.6580074-1	3.7456957-1	3.8321760-1	3.9185040-1
0.40	3.3387735-1	3.4434700-1	3.5459969-1	3.6536992-1
0.50	3.0964290-1	3.2076660-1	3.3258287-1	3.4450276-1
0.60	2.9045483-1	3.0300491-1	3.1575650-1	3.2865635-1
0.70	2.7690455-1	2.9005339-1	3.0338995-1	3.1696087-1
0.80	2.6721900-1	[illegible]	2.9447173-1	3.0850895-1
0.90	2.6037528-1	2.7410811-1	2.8814821-1	3.0249710-1
1.00	2.5559751-1	2.6946667-1	2.8371045-1	2.9826977-1

z	r = 0.78	r = 0.80	r = 0.82	r = 0.84
0.10	4.6387237-1	4.6703973-1	4.7030251-1	4.7343576-1
0.20	4.3008239-1	4.3622727-1	4.4238696-1	4.4855014-1
0.30	4.0048409-1	4.0915730-1	4.1782284-1	4.2655458-1
0.40	3.7594995-1	3.8659609-1	3.9732296-1	4.0814254-1
0.50	3.5653403-1	3.6869201-1	3.8098457-1	3.9342715-1
0.60	3.4173197-1	3.5499169-1	3.6844428-1	3.8209957-1
0.70	3.3076794-1	3.4481210-1	3.5909937-1	3.7363696-1
0.80	3.2282216-1	3.3741524-1	3.5229267-1	3.6745958-1
0.90	3.1715606-1	3.3213004-1	3.4741971-1	3.6302963-1
1.00	3.1316599-1	[illegible]	[illegible]	[illegible]

z	r = 0.86	r = 0.88	r = 0.90	r = 0.92
0.10	4.7666440-1	4.7985916-1	4.8309011-1	4.8637197-1
0.20	4.5474216-1	4.6097592-1	4.6726496-1	4.7362259-1
0.30	4.3534898-1	4.4422044-1	4.5318490-1	4.6225914-1
0.40	4.1907263-1	4.3012028-1	4.4132439-1	4.5267736-1
0.50	4.0603168-1	4.1681165-1	4.3178109-1	4.4495455-1
0.60	[illegible]	[illegible]	[illegible]	[illegible]
0.70	[illegible]	[illegible]	4.1883129-1	4.3445930-1
0.80	3.8292147-1	3.9868458-1	4.1475571-1	[illegible]
0.90	[illegible]	[illegible]	[illegible]	[illegible]
1.00	3.7615872-1	3.9277210-1	4.0973506-1	4.2706099-1

TABLE XI

Axial Velocity K_z For A Point Source

z	r = 0.00	r = 0.10	r = 0.20	r = 0.30
0.10	[illegible]	1.9550269-1	[illegible]	[illegible]
0.20	[illegible]	[illegible]	[illegible]	[illegible]
0.30	5.7069526	[illegible]	3.4967665	2.2050881
0.40	[illegible]	[illegible]	[illegible]	[illegible]
0.50	[illegible]	2.2461094	[illegible]	[illegible]
0.60	1.8081170	1.7734368	[illegible]	[illegible]
0.70	1.5186336	[illegible]	1.4205783	1.3085447
0.80	1.3332767	1.3166195	1.2706089	[illegible]
0.90	1.2175246	1.2079299	1.1782991	1.1391722
1.00	[illegible]	[illegible]	1.1204499	1.0944392

z	r = [illegible]	r = [illegible]	r = [illegible]	r = [illegible]
0.10	[illegible]	7.9504455-2	[illegible]	[illegible]
0.20	[illegible]	[illegible]	1.1610646	1.0554444
0.30	1.7022192	1.4475914	1.3402296	[illegible]
0.40	[illegible]	[illegible]	1.3506602	[illegible]
0.50	[illegible]	[illegible]	[illegible]	[illegible]
0.60	[illegible]	[illegible]	[illegible]	[illegible]
0.70	[illegible]	1.1876782	1.1654736	[illegible]
0.80	[illegible]	[illegible]	1.1179280	[illegible]
0.90	1.1119466	1.0920839	1.0887461	[illegible]
1.00	1.0763608	1.0690851	1.0575051	1.0511895

z	r = [illegible]	r = [illegible]	r = [illegible]	r = [illegible]
0.10	5.6694788-1	5.1800439-1	4.6658059-1	[illegible]
0.20	[illegible]	[illegible]	[illegible]	7.5996089-1
0.30	[illegible]	[illegible]	[illegible]	[illegible]
0.40	1.2126146	1.1518909	1.0960068	1.0450995
0.50	[illegible]	[illegible]	[illegible]	[illegible]
0.60	1.1615930	1.1319597	1.1025380	[illegible]
0.70	1.1226476	1.1021778	1.0824160	[illegible]
0.80	1.0895478	1.0757821	1.0623658	[illegible]
0.90	1.0638785	1.0546288	1.0455509	1.0366756
1.00	[illegible]	[illegible]	[illegible]	[illegible]

z	r = [illegible]	r = [illegible]	r = [illegible]	r = [illegible]
0.10	[illegible]	[illegible]	[illegible]	[illegible]
0.20	[illegible]	[illegible]	[illegible]	[illegible]
0.30	[illegible]	[illegible]	[illegible]	[illegible]
0.40	[illegible]	[illegible]	[illegible]	[illegible]
0.50	1.0395730	1.0062025	9.7538794-1	[illegible]
0.60	1.0492256	1.0247508	1.0017238	9.8009707-1
0.70	1.0452910	1.0277934	1.0112354	[illegible]
0.80	[illegible]	[illegible]	[illegible]	[illegible]
0.90	[illegible]	[illegible]	1.0115575	[illegible]
1.00	[illegible]	[illegible]	[illegible]	[illegible]

TABLE II - [illegible]

Z	r = 0.60	r = 0.70	r = 0.80	r = 0.90
0.10	[illegible]	[illegible]	[illegible]	[illegible]
0.20	[illegible]	4.5133[illegible]-1	3.[illegible]9177-1	[illegible].5119315-1
0.30	[illegible]	[illegible]	[illegible]4772128-1	[illegible]2401066-1
0.40	[illegible]	[illegible]	[illegible]	[illegible]
0.50	[illegible]	[illegible]	[illegible]	[illegible]
0.60	9.[illegible]-1	8.0[illegible]-1	[illegible]	[illegible]
0.70	9.[illegible]-1	9.[illegible]-1	[illegible]	[illegible]
0.80	9.[illegible]-1	[illegible]	9.20902[illegible]-1	9.0158611-1
0.90	[illegible]	[illegible]996858-1	9.4587765-1	9.3134870-1
1.00	[illegible]	[illegible]014509-1	9.[illegible]-1	9.5841557-1

Z	r = 1.00
0.10	1.[illegible]-1
0.20	[illegible]
0.30	4.0[illegible]-1
0.40	[illegible]
0.50	[illegible]
0.60	[illegible]
0.70	[illegible]
0.80	[illegible]
0.90	[illegible]
1.00	[illegible]

TABLE III

Radial Velocity ϕ_r for A Point Source

z	r = 0.10	r = 0.20	r = 0.30	r = 0.40
0.10	1.[illegible]+1	[illegible]	[illegible]	[illegible]
0.20	[illegible]	4.3292149	[illegible]	2.[illegible]5
0.30	1.[illegible]	2.0582[illegible]	1.[illegible]511[illegible]	1.[illegible]69516
0.40	6.[illegible]-1	1.0[illegible]6858[illegible]	1.[illegible]	9.6057603-1
0.50	3.[illegible]-1	5.73[illegible]-[illegible]	6.5616260	6.2706175-1
0.60	2.9136[illegible]-1	3.[illegible]-2	4.0419557	4.0[illegible]-1
0.70	1.1321095-1	2.0272[illegible]-2	2.5[illegible]86998	2.6861[illegible]-1
0.80	6.9820[illegible]-2	1.2718962-1	1.6388[illegible]10	1.775[illegible]-1
0.90	4.[illegible]-2	8.165[illegible]-2	1.06[illegible]	1.[illegible]540-1
1.00	2.[illegible]-2	5.3500[illegible]-2	7.0637967	7.9119285-2

z	r = 0.48	r = 0.50	r = 0.52	r = 0.54
0.10	2.7211176	1.67772[illegible]	1.590[illegible]	1.3996[illegible]5
0.20	1.7777500	1.[illegible]358	1.2[illegible]	1.1939977
0.30	1.2876580	1.06764.56	9.99[illegible]-1	9.94[illegible]-1
0.40	8.8670940-1	7.6[illegible]62-1	7.30[illegible]-1	6.914[illegible]-1
0.50	5.[illegible]-1	5.[illegible]72-1	5.157[illegible]-1	4.[illegible]-1
0.60	3.[illegible]6205-1	3.[illegible]57[illegible]-1	3.5730097-1	3.[illegible]-1
0.70	2.[illegible]679-1	2.5[illegible]-1	2.[illegible]51[illegible]-1	2.[illegible]-1
0.80	1.7707252-1	1.7[illegible]209-1	1.6738158-1	1.634[illegible]-1
0.90	1.1880097-1	1.1[illegible]064-1	1.1105272-1	1.1175[illegible]-1
1.00	7.9979920-2	7.8715067-2	7.7655287-2	7.6298[illegible]20-2

z	r = 0.56	r = 0.58	r = 0.60	r = 0.62
0.10	1.[illegible]	[illegible]	1.[illegible]	[illegible]-1
0.20	1.[illegible]	1.0[illegible]	9.[illegible]-1	[illegible]-1
0.30	8.[illegible]-1	8.[illegible]-1	7.5716750-1	7.0[illegible]37-1
0.40	6.[illegible]15-1	6.1510801-1	5.7707860-1	5.[illegible]-1
0.50	4.7059469-1	4.[illegible]515-1	4.2[illegible]-1	4.0061087-1
0.60	3.9159227-1	3.1766205-1	3.0927660-1	2.[illegible]-1
0.70	2.3042260-1	2.2212954-1	2.1390530-1	2.0401561-1
0.80	1.[illegible]9-1	1.5[illegible]-1	1.[illegible]561-1	1.[illegible]-1
0.90	1.0806011-1	1.0[illegible]-1	1.0[illegible]-1	[illegible]
1.00	7.[illegible]-2	7.27[illegible]-2	7.0[illegible]687-2	6.8[illegible]-2

z	r = 0.64	r = 0.66	r = 0.68	r = 0.70
0.10	9.0607[illegible]96-1	8.2667940-1	7.5219589-1	6.8208209-1
0.20	7.9505739-1	7.2995428-1	6.6866952-1	6.1094466-1
0.30	6.5196722-1	6.0[illegible]91-1	5.5583132-1	5.1079617-1
0.40	5.0593749-1	4.7101125-1	4.3741618-1	4.0446384-1
0.50	3.[illegible]-1	3.[illegible]-1	3.[illegible]-1	3.[illegible]-1
0.60	2.[illegible]32951-1	2.5792949-1	2.4238757-1	2.2561[illegible]-1
0.70	1.9432214-1	1.8420252-1	1.7395039-1	1.6387551-1
0.80	1.3696569-1	1.[illegible]-1	1.2[illegible]-1	1.1[illegible]-1
0.90	9.4852798-2	9.0[illegible]-2	8.6057[illegible]-2	8.1326693-2
1.00	6.[illegible]-2	6.2765372-2	5.9[illegible]-2	5.[illegible]-2

[illegible]

TABLE III - continued

Z	r = 0.72	r = 0.74	r = 0.76	r = 0.78
0.10	6.1884812-1	5.5980414-1	5.0502757-1	4.5353166-1
0.20	5.5685676-1	5.0478804-1	[illegible]	[illegible]
0.30	4.6746537-1	4.2794004-1	3.8668871-1	3.4762675-1
0.40	3.7237541-1	[illegible]	[illegible]	[illegible]
0.50	2.8459778-1	2.6204496-1	2.3979066-1	2.1785773-1
0.60	2.1082388-1	1.9498841-1	[illegible]	[illegible]
0.70	1.5260354-1	1.4167820-1	1.3068756-1	1.1951815-1
0.80	1.0863690-1	1.0116473-1	9.3525135-2	[illegible]
0.90	7.6406992-1	7.1922046-2	6.4097792-2	6.0758596-2
1.00	5.3275812-1	4.9821220-2	[illegible]	[illegible]

Z	r = 0.80	r = 0.82	r = 0.84	r = 0.86
0.10	4.0434655-1	3.5678846-1	[illegible]	2.6581820-1
0.20	3.6463446-1	3.2204625-1	2.8118476-1	2.4187334-1
0.30	3.1101888-1	2.7560015-1	[illegible]	[illegible]
0.40	2.5282101-1	2.2425199-1	[illegible]	1.7082551-1
0.50	1.9626139-1	1.7581575-1	[illegible]	[illegible]
0.60	1.4769469-1	1.3210836-1	1.1665457-1	1.0135464-1
0.70	1.0875522-1	[illegible]	[illegible]	[illegible]
0.80	7.7775535-2	[illegible]	[illegible]	[illegible]
0.90	5.5315584-2	4.9776824-2	4.4228971-2	[illegible]
1.00	3.8821535-2	3.4997461-2	3.1120496-2	[illegible]

Z	r = 0.88	r = 0.90	r = 0.92
0.10	[illegible]	[illegible]	[illegible]
0.20	[illegible]	[illegible]	[illegible]
0.30	[illegible]	[illegible]	[illegible]
0.40	[illegible]	[illegible]	[illegible]
0.50	[illegible]	[illegible]	[illegible]
0.60	[illegible]	[illegible]	[illegible]
0.70	6.3820190-2	5.2046008-2	[illegible]
0.80	4.6262297-2	3.6355151-2	[illegible]
0.90	3.3007346-2	2.7393008-2	[illegible]
1.00	[illegible]	[illegible]	[illegible]

APPENDIX

Solution of Equation 1 for $K = j^2$

Referring to Eq. 1, and considering the case of $K = j^2$, one obtains,

$$Z_{zz} + j^2 Z = 0$$

$$R_{rr} + \frac{R_r}{r} - j^2 R = 0$$

The solutions to the above equations are well known, and can be written as,

$$Z = F_1 \cos jz + F_2 \sin jz$$

$$R = G_1 I_0(jr) + G_2 K_0(jr)$$

The third solution to Eq. 1 is,

$$\phi = (F_1 \cos jz + F_2 \sin jz)\left(G_1 I_0(jr) + G_2 K_0(jr)\right)$$

where (I_0) is the modified Bessel function of the first kind of zero order and (K_0) is the modified Bessel function of the second kind of zero order.

It is of interest to determine whether the boundary conditions for a point source can be satisfied with the solution above. Examining condition I, it is apparent that the above solution is periodic so that a series solution cannot represent this condition.

A series solution of theoretical interest can be found for the above solution that represents a flow within a duct of constant diameter. To satisfy the condition of zero radial velocity at the duct wall, one can write,

$$R = I_0(jr)K_1(j) - K_0(jr)I_1(j)$$

where (I_1) is the modified Bessel function of the first kind of the first order, and (K_1) is the modified Bessel function of the second kind of the first order. The form of the solution is a result of the fact that neither (I_1) or (K_1) have real roots except at $r = 0$, (Ref. 6), so that a combination of the two functions is required to obtain a function that is zero for some value of (r), i.e., satisfies the condition of zero radial velocity at the duct wall.

A flow model, which can be satisfied by the solution obtained above, is that of a periodic distribution of sources and sinks along the duct axis. Considering that the sources and sinks are spaced (L) radii apart, and that flow symmetry exists at these points, one has,

$$(\phi_s)_{r=1} = 0 = \left[I_0(jr)K_1(j) - K_0(jr)I_1(j)\right] \; j(F_2 \cos jz - F_1 \sin jz)$$

where $s = 1, 2, 3, 4, \ldots$

Thus, $F_2 = 0$, and $j = s\pi/b$, so that one has,

$$\phi = \sum_{n=1}^{\infty} A_n \quad (I_0(j_n r)K_1(j_n) - K_0(j_n r)I_1(j_n)) \quad \cos j_n z \qquad \text{(I-1)}$$

To determine the coefficients, (A_n), Gauss's theorem can be applied. With alternating sources and sinks of equal strength, one can use this theorem midway between them, i.e., when

$$z = \frac{2u-1}{2} b$$

Differentiating Eq. I-1, with respect to (z), multiplying through by

$$\left(r \sin j_n \frac{2u-1}{2} b\right),$$

and integrating the result at

$$z = \frac{2u-1}{2} b,$$

yields for the left hand side,

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 (\phi_z)_{z=\frac{2u-1}{2}b} \sin j_p \frac{2u-1}{2} b \; r\,dr\,d\theta = \pm m$$

The above integral is valid when (p) is odd, $(j_p = p\pi/b)$.

Thus, let $p = 2u-1$, where $u = 1, 2, 3, 4, \ldots$ The (+) sign holds for odd values of (u), the (-) sign for even values. The value of the above integral is $(\pm m)$ as the flow of each source splits, $(2\pi m)$ going in either direction.

For the right hand side (see above),

$$\pm m = -A_p j_p K_1(j_p) \int_0^1 r I_0(j_p r) \sin^2 j_p \frac{2u-1}{2} b \, dr$$

$$+ A_p j_p I_1(j_p) \int_0^1 r K_0(j_p r) \sin^2 j_p \frac{2u-1}{2} b \, dr$$

Performing the integration, noting that

$$\lim_{r \to 0} r j_p K_1(j_p r) = 1$$

yields,

$$m = \frac{A_p I_1(j_p)}{j_p}$$

The resultant expression for the velocity potential is,

$$\phi = z \sum_{n=1}^{\infty} j_n \left[\frac{I_0(j_n r) K_1(j_n) - K_0(j_n r) I_1(j_n)}{I_1(j_n)} \right] \cos j_n z$$

where $j_n = (2n-1) \pi b$.

APPENDIX II

Calculation of Special Body Shapes

Body A

The body coordinates and source locations chosen are:

	x	y		x	y
a_1	−.59	.30	a_5	−.165	.65
a_2	−.49	.40	a_6	0	.70
a_3	−.365	.50	a_7	.19	.75
a_4	−.26	.60	a_8	.62	—

Instead of specifying eight body coordinates, seven are specified along with the condition that $S_8 = .500$. Eight linear equations in eight unknowns can now be set up. To do this requires the determination of 64 coefficients. Taking the first body coordinate, and determining the value of the stream function there using Eq. 28,

$$0 = .045 - a_1F^*_{11} - a_2F^*_{21} - a_3F^*_{31} - a_4F^*_{41} - a_5F^*_{51} - a_6F^*_{61} - a_7F^*_{71} - a_8F^*_{81}$$

The values of F^* are obtainable from Table I. For F^*_{11}, $|m| = 0$, so that $F^*_{11} = 1/2$. For F^*_{21}, $|m| = .10$, so that $F^*_{21} = .3456$.

Continuing in this manner, all the coefficients for the eight equations can be quickly found. Setting the result up in matrix form

$$\begin{pmatrix} .500 & .346 & .266 & .177 & .128 & .084 & .062 & .048 \\ .614 & .500 & .425 & .317 & .249 & .151 & .118 & .087 \\ .638 & .555 & .500 & .410 & [illegible] & [illegible] & [illegible] & [illegible] \\ .669 & .594 & .570 & .500 & .436 & .341 & .266 & .190 \\ .690 & .640 & .620 & .554 & .500 & .409 & .304 & .297 \\ .696 & .674 & .655 & .600 & .576 & .500 & .419 & .288 \\ .691 & .680 & .670 & .652 & .622 & .575 & .500 & .353 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ S_8 \end{pmatrix} = \begin{pmatrix} .045 \\ .080 \\ [illegible] \\ .160 \\ .211 \\ .245 \\ .281 \\ .500 \end{pmatrix}$$

The above matrix was solved by hand, by diagonalizing it. The results were:

$a_1 = .243$ $a_3 = -.567$ $a_5 = .977$ $a_7 = -.102$

$a_2 = -.214$ $a_4 = .296$ $a_6 = -.232$ $a_8 = .159$

To obtain the velocities, Tables II and III can be used. For the axial velocity at $z = 0$, $r = 1$, one has by Eq. 30,

$$(\phi_z)_{z=0} = 1 + s_1(\phi_z^\circ)_1 + s_2(\phi_z^\circ)_2 + s_3(\phi_z^\circ)_3 + s_4(\phi_z^\circ)_4 + s_5(\phi_z^\circ)_5 + s_6(\phi_z^\circ)_6$$
$$- s_7(\phi_z^\circ)_7 - s_8(\phi_z^\circ)_8$$

To find $(\phi_z^\circ)_1$, the values $z = .53$, $r = 1$ from Table II correspond to $(\phi_z^\circ)_1 = .780$. The values of (s) are known from the solution of matrix, and the remaining velocity terms can be found from the tables. The radial velocities are calculated in the same manner, using Table III.

Body 2

The body coordinates and source locations chosen are:

	z	r		z	r
s_1	-.90	.10	s_5	-.35	.50
s_2	-.71	.20	s_6	-.22	.60
s_3	-.57	.30	s_7	0	.70
s_4	-.45	.40	s_8	.20	—

The sum of the source strengths was fixed at .506. The results of the calculation are:

$s_1 = .009$	$s_5 = .116$
$s_2 = .018$	$s_6 = .294$
$s_3 = .071$	$s_7 = .204$
$s_4 = .057$	$s_8 = .180$

APPENDIX III

[illegible]

The following conditions are prescribed:

Body coordinates:

z	r
-.45	.40
0	.70
.52	.60

Velocity distribution at z = 0:

$$(\phi_s)_{r=1} = 1.2(\phi_s)_{r=.7}$$

$$(\phi_s)_{r=.9} = 1.133(\phi_s)_{r=.7}$$

$$(\phi_s)_{r=.8} = 1.067(\phi_s)_{r=.7}$$

The total source strength will be .596.

There are seven conditions specified, requiring seven sources and sinks. The location of these sources and sinks are:

	z		z
S_1	-.71	S_5	-.20
S_2	-.57	S_6	0
S_3	-.46	S_7	.20
S_4	-.35		

Note that the arrangements of sources departs from the method used in Appendix II, where they were located at the same (z) coordinate as the body points.

The set up of the three equations for the body coordinates follows the same approach used in Appendix II. The equations for the velocity conditions are set up as follows:

[illegible] Eq. 30, 90

[illegible]

at x = 0, r = 1, and x = 0, r = .7, and equating the two expressions in the proper ratio, and combining like terms, [illegible] the necessary coefficients. This procedure is repeated three times, once for each velocity condition. Thereupon, the final system of equations may be written as,

$$\begin{pmatrix} .754 & .634 & .505 & .386 & .267 & .155 & .114 \\ .724 & .704 & .678 & .651 & .600 & .500 & .410 \\ .679 & .674 & .671 & .666 & .658 & .634 & .500 \\ .260 & .289 & .287 & .270 & .211 & 0 & -.240 \\ .192 & .215 & .224 & .216 & .169 & 0 & -.158 \\ .108 & .122 & .129 & .128 & .100 & 0 & -.099 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \end{pmatrix} = \begin{pmatrix} .000 \\ .245 \\ .920 \\ -.200 \\ -.159 \\ -.067 \\ .506 \end{pmatrix}$$

The resultant values for the source and sink strengths are:

$S_1 = 5.691$ $S_3 = 5.660$ $S_5 = 25.173$ $S_7 = -.525$

$S_2 = .497$ $S_4 = 17.385$ $S_6 = 8.348$

Having found the source and sink strengths, the complete body shape can be found by utilizing Eq. 28. The results are shown in Fig. 10.

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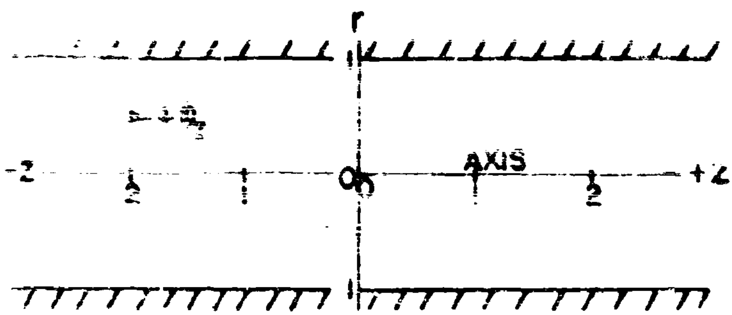


FIGURE 1. COORDINATE SYSTEM USED

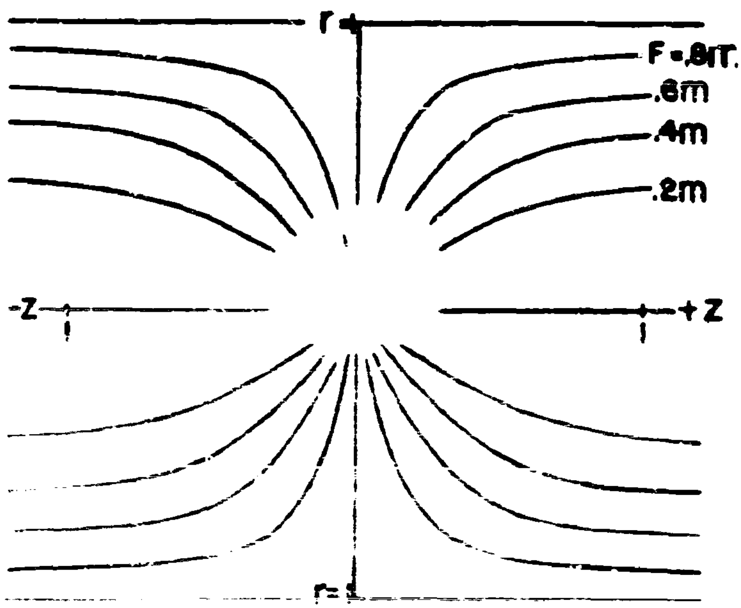


FIGURE 2. STREAMLINES OF A POINT SOURCE

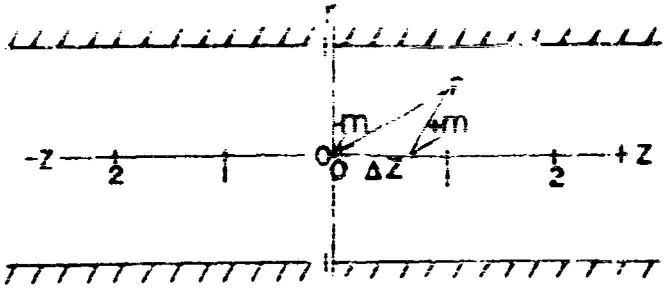


FIGURE 3. DERIVATION OF THE VELOCITY POTENTIAL FOR A DOUBLET

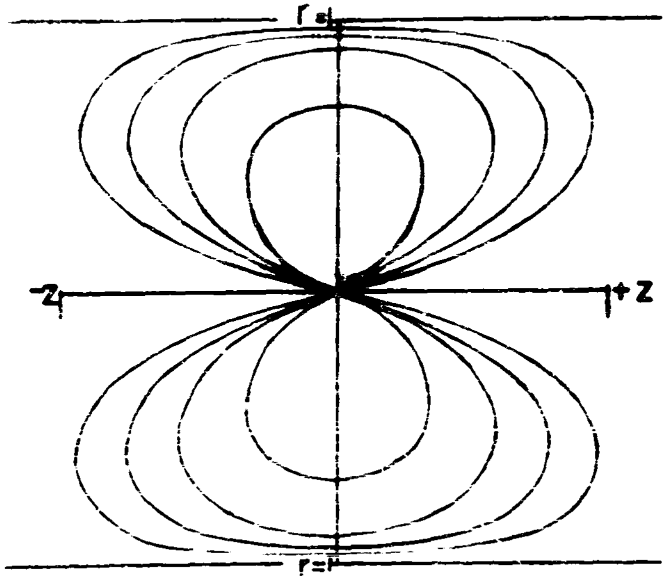


FIGURE 4. STREAMLINES OF A DOUBLET

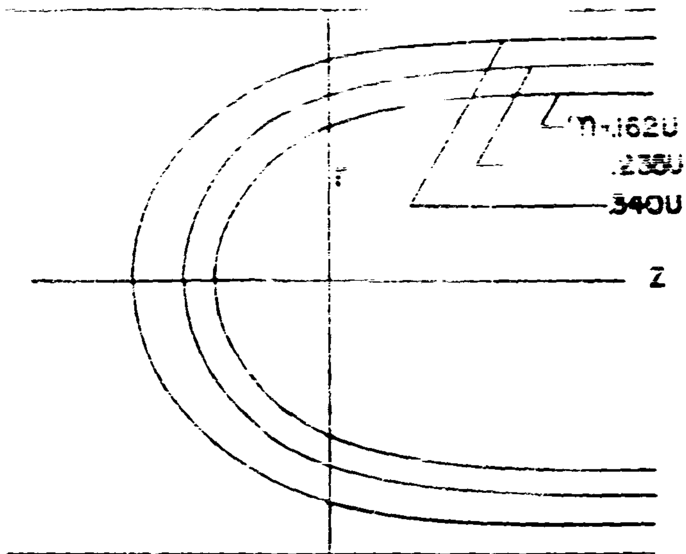


FIGURE 5. POINT SOURCES IN A UNIFORM FLOW

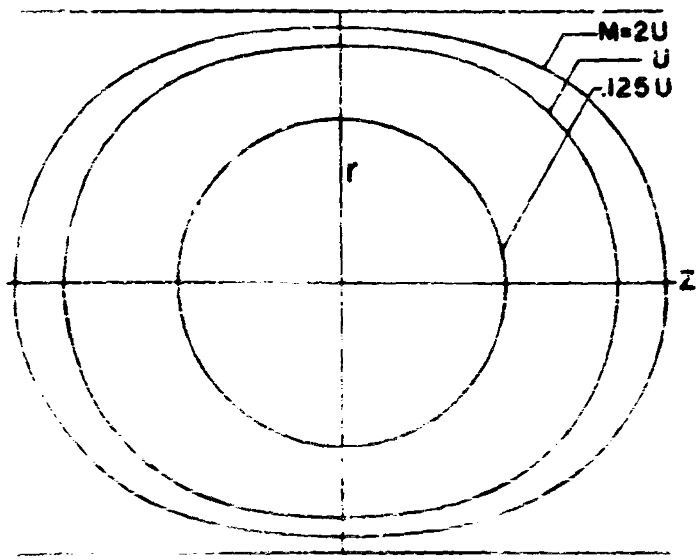


FIGURE 6. DOUBLETS IN A UNIFORM FLOW

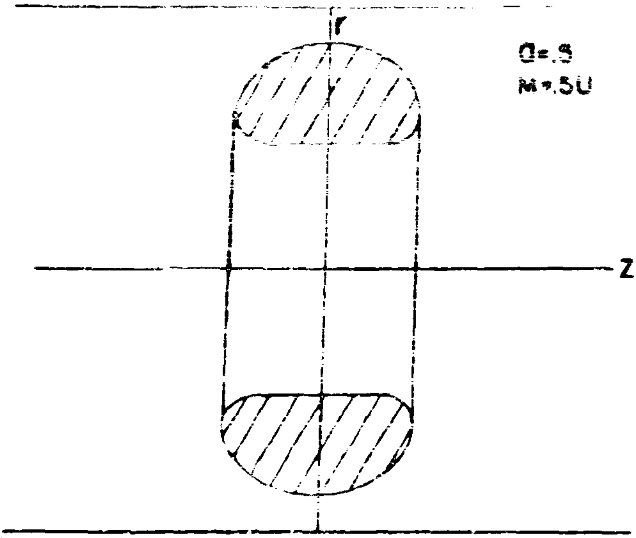


FIGURE 7 RING DOUBLET IN A UNIFORM FLOW

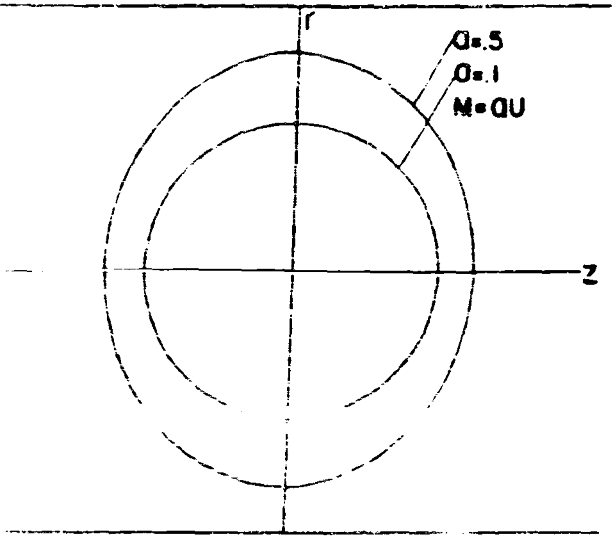
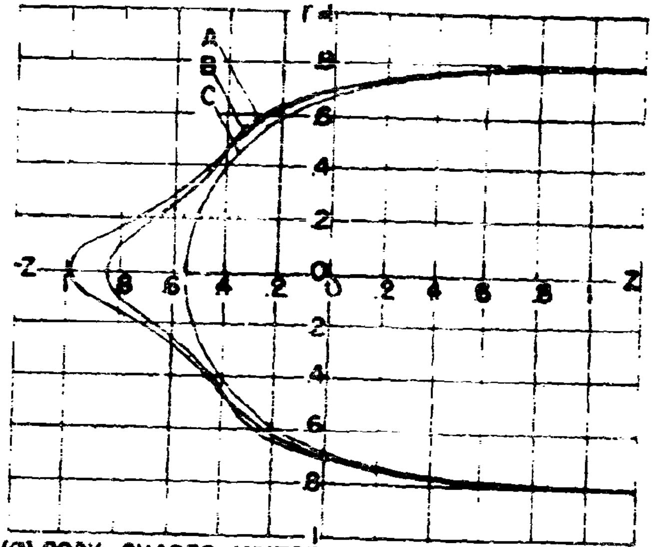
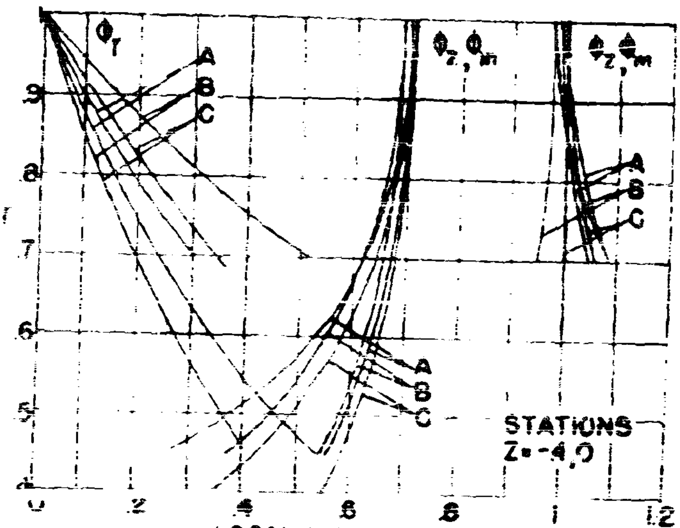


FIGURE 8 DISK DOUBLET IN A UNIFORM FLOW

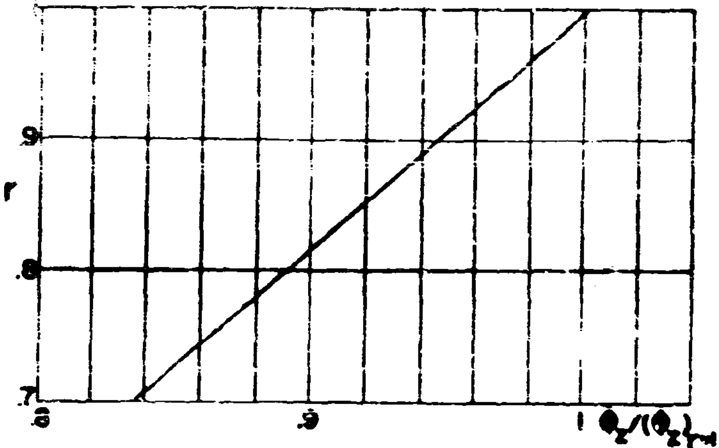


(a) BODY SHAPES INVESTIGATED

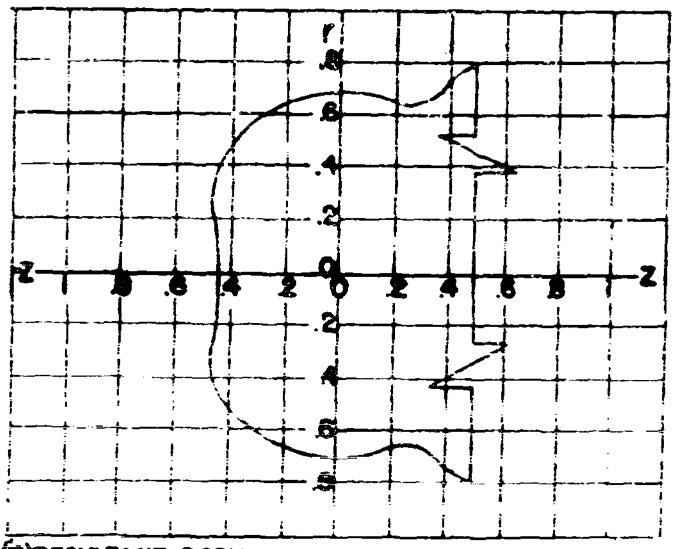


(b) VELOCITY DISTRIBUTIONS FOR BODIES (A,B,C)

FIGURE 9 ARBITRARY BODY SHAPES INVESTIGATED

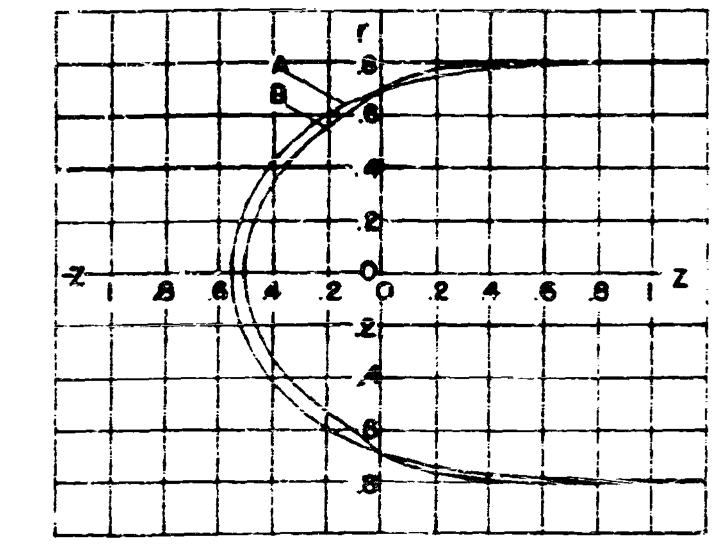


(a) PRESCRIBED VELOCITY DISTRIBUTION AT Z=0

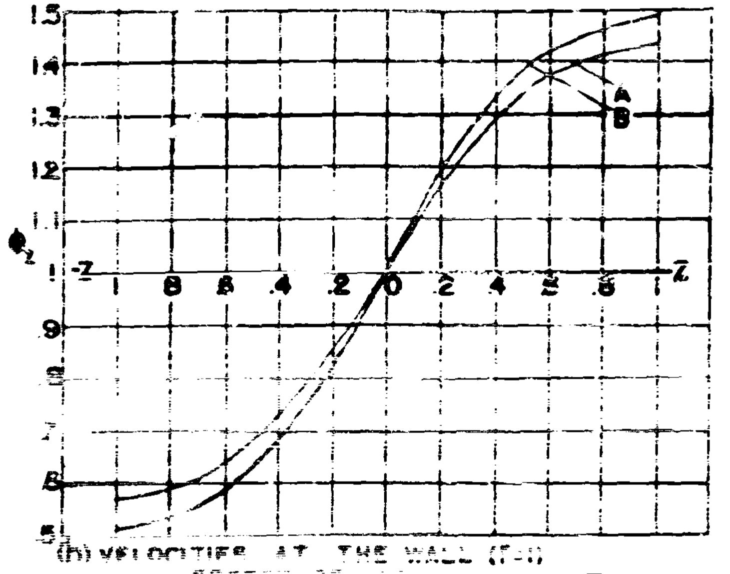


(b) RESULTANT BODY

FIGURE 10. RESULTANT BODY FOR A PRESCRIBED VELOCITY DISTRIBUTION



(a) LOCAL EFFECT OF A RING SOURCE



(b) VELOCITIES AT THE WALL (r=1)

FIGURE 11 EFFECT OF LOCAL CURVATURE